PMM U.S.S.R.,Vol.51,No.6,pp.758-761,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

A DIVERGENT SYSTEM OF NON-STATIONARY EQUATIONS OF MOTION OF VISCOELASTIC MEDIA IN EULER COORDINATES*

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A system of divergent equations of non-stationary motions of viscoelastic media is presented. It is shown that for continuous flows it is equivalent to the well-known system of equations in /1/. Divergent equations are preferable, for instance, from the viewpoint of their utilization in calculational algorithms. On the basis of the divergent forms obtained, relationships at discontinuities are analysed.

We introduce a fixed orthonormal system of coordinates with basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The location of an element of a continuous medium in the initial undeformed state and the running time is characterized by the vectors $\mathbf{X} = (X_1, X_2, X_3)$ and $\mathbf{x} = (x_1, x_2, x_3)$, where the coordinates are referred to the basis selected.

The relation $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ between the running and initial locations of a point determined by the motion of the medium results in a relationship for changing the shape of the element of the continuous medium

$$d\mathbf{x} = \mathbf{A}\, d\mathbf{X}; \quad \mathbf{A} = \| A_{ik} \|, \quad A_{ik} = (\partial x_i/\partial X_k)\ X_j$$

Since the consideration is performed in orthogonal coordinates, no distinction is made below between the covariant and contravariant subscripts. Moreover, summation is understood to be over identical indices in the relationships produced later.

We introduce a concept often utilized in research. The Cauchy and Almansi strain tensors /2/ are a measure of the deformation of the continuous medium.

$$\varepsilon = (\mathrm{E} - \mathrm{G})/2, \quad \varepsilon_1 = (\mathrm{G}_1 - \mathrm{E})/2, \quad \mathrm{G} = \mathbf{B}^*\mathbf{B}, \quad \mathrm{G}_1 = \mathbf{A}^*\mathbf{A}$$

(E is the unit matrix, $\mathbf{B} = \mathbf{A}^{-1}$, $\mathbf{F}^*$ is the transpose and $\mathbf{F}^{-1}$ is the inverse of the matrix F).

If $ds_0^2 = (d\mathbf{X},\ d\mathbf{X})$ and $ds^2 = (d\mathbf{x},\ d\mathbf{x})$ are squares of the undeformed and deformed segment lengths connecting two nearby points of the medium, then $ds_0^2 - ds^2 = -2\ (\varepsilon d\mathbf{x},\ d\mathbf{x}) = -2\ (\varepsilon_1 d\mathbf{X}, d\mathbf{X})$ ((**a**, **b**) is the scalar product of the vectors **a** and **b**), $ds^2 = (d\mathbf{x}, d\mathbf{x}) = (\mathrm{G}_1\, d\mathbf{X}, d\mathbf{X})$, $ds_0^2 = (\mathrm{G}\, d\mathbf{x}, d\mathbf{x}) = (d\mathbf{X},\ d\mathbf{X})$. (By definition, ε and ε_1 characterize the change in the distance between points of the continuous medium $\varepsilon, \varepsilon_1, \mathrm{G}, \mathrm{G}_1$ are symmetric tensors.

The affinor $\mathbf{A}$ can be represented in the form /3/: $\mathbf{A} = \mathbf{R}\mathbf{F}_1 = \mathbf{F}\mathbf{R}$, where $\mathbf{F}, \mathbf{F}_1$ are symmetric and positive-definite matrices, and R is an orthogonal matrix. Here, obviously $\mathbf{F}_1^2 = \mathrm{G}_1$, $\mathbf{F}^2 = \mathrm{G}$.

From the definition of the affinor $\mathbf{A}$ and the velocity vector of the element of the continuous medium $\mathbf{u} = (u_1, u_2, u_3)$ we have

$$\frac{\partial A_{ij}}{\partial t} + u_k \frac{\partial A_{ij}}{\partial x_k} = \frac{\partial u_i}{\partial x_k} A_{kj}$$

It follows from the law of conservation of mass that the density is $\rho = \rho_0/\det \mathbf{A}$ (the subscript zero denotes the density of the medium in the initial state). Multiplying both sides of the equation for $\mathbf{A}$ by ρ and utilizing the equation of continuity, we conclude that the following divergent equations for $\mathbf{A}$ are correct (we see by direct calculations that $\partial \rho A_{kj}/\partial x_k = 0$):

$$\frac{\partial \rho A_{ij}}{\partial t} + \frac{\partial \rho u_k A_{ij}}{\partial x_k} = \frac{\partial \rho u_i A_{kj}}{\partial x_k} \tag{1}$$

which in combination with the divergent equations expressing the laws of conservation of momentum and energy

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_k} \rho u_i u_k - \frac{\partial \sigma_{ik}}{\partial x_k} = 0 \tag{2}$$

$$\partial \left(\rho\left(e + \left(\frac{\mathbf{u}, \mathbf{u}}{2}\right)\right)\right) + \operatorname{div}\left(\rho \mathbf{u} \left(e + \frac{(\mathbf{u}, \mathbf{u})}{2}\right)\right) - \frac{\partial \sigma_{ik} u_i}{\partial x_k} = 0$$

*Prikl.Matem.Mekhan.,51,6,984-988,1987

form the system of equations of motion of a continuous medium.

To conserve the unity of the discussion, we first consider a non-linear elastic medium. The equations of motion for it are closed by the equation of state $e = e(\mathbf{A}, s)$ that yields the specific internal energy as a function of $\mathbf{A}$ and the specific energy s, where the first law of thermodynamics that expresses the connection between the internal energy and entropy increments with the work by elastic displacements

$$de = \rho^{-1}\boldsymbol{\sigma}\circ d\mathbf{A}\cdot\mathbf{A}^{-1} + T ds$$

is valid, ($\boldsymbol{\sigma} = \|\sigma_{ik}\|$ is the stress tensor, T is the temperature, $\mathbf{P}\circ\mathbf{Q} = P_{ik}Q_{ik}$ is the convolution of two matrices). Hence $\sigma = \rho\,(\partial e/\partial \mathbf{A})_s A^*$. Considering e to be a function of the strain tensor $\boldsymbol{\varepsilon}$, the expression for σ can be converted to the form $\boldsymbol{\sigma} = \rho\,(\mathbf{E} - 2\boldsymbol{\varepsilon})\,(\partial e/\partial \boldsymbol{\varepsilon})_j$. This last relationship is the Murnaghan formula /4/.

The relationships

$$\begin{gathered}
-[\rho A_{ij}]\,D + [\rho u_n A_{ij}] - [\rho u_i A_{lj} n_l] = 0 \\
-[\rho u_i]\,D + [\rho u_i u_n] - [\sigma_{ik} n_k] = 0 \\
-[\rho\,(e + (u, u)/2)]\,D + [\rho u_n\,(e + (u, u)/2) + \sigma_{ik} u_i n_k] = 0 \\
D = -\frac{S_t}{|\nabla S|}, \quad \mathbf{n} = \frac{\nabla S}{|\nabla S|}, \quad S_t = \left.\frac{\partial S}{\partial t}\right|_{x_j}, \quad S_{x_k} = \left.\frac{\partial S}{\partial x_k}\right|_{t, x_j}
\end{gathered}$$

must be satisfied on the surface of discontinuity $S(x, t)$, where D is the velocity of the surface of discontinuity, $\mathbf{n}$ is the normal to it, and $[f]$ is the difference between values of the quantity f on opposite sides of S. The gradient is $\nabla S = (S_{x_1}, S_{x_2}, S_{x_3})$.

The conditions for the momentum and energy on the discontinuity are well-known. The relations obtained in a formal manner from the divergent equations of motion for $\mathbf{A}$ agree on the shockwaves with the kinematic relationships /5/ which express continuity of the displacements. To see this, we rewrite them in a system of coordinates for which the direction of one of its axes agrees with $\mathbf{n}$ (to be specific, let this be the x_1 axis)

$$\begin{gathered}
[\rho A_{1l}]\,D = 0, \quad l = 1, 2, 3 \\
-j\,[A_{ki}] = [u_k]\,\rho^1 A_{1i}{}^1, \quad k = 2, 3, \quad i = 1, 2, 3 \\
j = \rho^1\,(D - u_1{}^1) = \rho^2\,(D - u_1{}^2)
\end{gathered}$$

(j is the mass flux through the discontinuity). The flow parameters in front of and behind the front will be denoted by the superscripts 1 and 2.

At the time $t = 0$ let an element of the medium with the Lagrange coordinates X occupy the position x_0 while the surface of discontinuity agrees with the plane $x_1 = 0$. At the time t the radius-vector x of the element of the medium being considered is determined by the equation

$$\begin{gathered}
x = x_0 + u_1 t\,\theta\,(t_1 - t) + u_1 t_1\,\theta\,(t - t_1) + \\
u_2\,(t - t_1)\,\theta\,(t - t_1) \\
\theta(\xi) = \begin{cases} 0, & \xi < 0, \\ 1, & \xi \geqslant 0, \end{cases} \quad t_1 = \frac{x_{01}}{D - u_1{}^1}
\end{gathered}$$

(t_1 is the time the discontinuity reaches this point of the medium).

Differentiating the expression for x with respect to X, we obtain relations the agree with the relations on the discontinuity that result from the divergent equations of motion.

The equations of conservation of momentum and energy in the system of coordinates selected in the manner mentioned above will yield the relations

$$\begin{gathered}
-j\,[u_i] = [\sigma_{i1}] \\
[e] = (\sigma_{i1}{}^1 + \sigma_{i1}{}^2)\,[A_{i1}]/(2\rho_1{}^1 A_{11}{}^1)
\end{gathered}$$

Unlike ideal gasdynamics, assignment of one of the parameters behind a shock front in the case of a non-linear elastic medium does not determine the remaining flow parameters uniquely. In general, three shock adiabatics correspond here to each initial state: one quasi-longitudinal and two quasitransverse /5/.

The following circumstance is also associated with the more complex structure of the equations.

We examine a fixed discontinuity relative to the substance. In this case $j = 0$ and it follows from the relationships presented above on the discontinuity that the velocities and stresses are continuous during passage across it. The density, entropy and the components $\mathbf{A}$ are generally discontinuous. In addition to the type of contact discontinuity considered, a situation of another kind is possible when the normal velocity and stress components are continuous during passage across the discontinuity, while the tangential components of the velocity undergo a discontinuity. As regards the tangential stress components, they should be given by using an additional model that describes the friction of the surfaces on each other. This case holds on the interface of media with different properties. In particular, if there is no such friction, the tangential stresses equal zero. In any case, six conditions must be satisfied on a contact discontinuity that ensure its evolution.

Turning to a consideration of a non-linear viscoelastic medium, we assume the representation $\mathbf{A} = \mathbf{A}_e\mathbf{A}_p$ to be valid, where $\mathbf{A}_e$ and $\mathbf{A}_p$ are, respectively, the elastic and plastic parts of the affinor $\mathbf{A}$, where the internal energy is a function of the entropy and the elastic part $\mathbf{A}_e$: $e = e(\mathbf{A}_e, s)$. Such an assumption means that the strain of an element of the continuous medium can be decomposed into two successive processes: first, plastic deformation and then elastic deformation from the new state obtained. As is seen from the sequel, plastic deformation in the model under consideration is understood to be the residual deformation of the element of the continuous medium after the adiabatic removal of the stress from it.

Let us write the first law of thermodynamics

$$de = \boldsymbol{\sigma} : d\mathbf{A}_e\mathbf{A}_e^{-1} + T ds$$

We will consider the rate of change of $\mathbf{A}_p$ during the motion of the element of the continuous medium to be a function of the running parameters of the state

$$\mathbf{A}_p^{\cdot} = \boldsymbol{\Phi}(\mathbf{A}_p, \mathbf{A}_e, s) \tag{3}$$

Here and henceforth, the dot denotes the Lagrange derivative ($f^{\cdot} = \partial f/\partial t + (\mathbf{u}, \nabla) f$). Combining (2) and (3) and the first law of thermodynamics, we obtain an expression for the entropy production

$$\rho T s^{\cdot} = \boldsymbol{\sigma} : \mathbf{A}_e\boldsymbol{\Phi}\mathbf{A}^{-1}$$

We hence conclude that the adiabatic process being performed for frozen plastic deformation is simultaneously isentropic. In combination with the equation of state and the assignment of the kinetics of plastic deformation, the function $\boldsymbol{\Phi}$, Eqs. (1) and (3) form a complete system of equations of motion of a viscoelastic medium. If both sides of (3) are multiplied by ρ and the equation of continuity is used, we obtain the divergent equation

$$\partial(\rho\mathbf{A}_p)/\partial t + \nabla(u\rho\mathbf{A}_p) = \rho\boldsymbol{\Phi} \tag{4}$$

It hence follows that the function $\mathbf{A}_p$ is continuous at the shock ($\mathbf{A}_p$ can undergo a discontinuity on the contact discontinuity).

By analogy with the tensors G and ε introduced earlier, we introduce the tensors

$$\mathrm{G}_e = \mathbf{B}_e{}^*\mathbf{B}_e, \quad \boldsymbol{\varepsilon}_e = (\mathrm{E} - \mathrm{G})/2, \quad \mathbf{P}_e = \mathbf{A}_e^{-1}$$

Since $\mathbf{B}_e^{\cdot}\mathbf{A}_e + \mathbf{B}_e\mathbf{A}_e^{\cdot} = 0$, we have the chain of equalities

$$\begin{gathered}\mathrm{G}_e^{\cdot} = \mathbf{B}_e^{\cdot *}\mathbf{B}_e + \mathbf{B}_e{}^*\mathbf{B}_e^{\cdot} = -\mathbf{B}_e{}^*(\mathbf{B}_p{}^*\mathbf{A}^{\cdot *} + \boldsymbol{\Phi}_1{}^*)\mathrm{G}_e - \\ \mathrm{G}_e(\mathbf{A}^{\cdot}\mathbf{B}_p + \boldsymbol{\Phi}_1)\mathbf{B}_e = -\mathbf{A}^{*-1}\mathbf{A}^{\cdot *}\mathrm{G}_e - \mathrm{G}_e\mathbf{A}^{\cdot}\mathbf{A}^{-1} - \boldsymbol{\Phi}_2{}^* - \boldsymbol{\Phi}_2 \\ \boldsymbol{\Phi}_1 = -\mathbf{A}_e\boldsymbol{\Phi}\mathbf{A}_p^{-1}, \quad \mathbf{B}_p = \mathbf{A}_p^{-1}, \quad \boldsymbol{\Phi}_2 = \mathrm{G}_e\boldsymbol{\Phi}_1\mathbf{B}_e\end{gathered}$$

The tensor ε_e is subject to the equations

$$\begin{gathered}\varepsilon_e^{\cdot} = {}^1/_2\,\mathbf{A}^{*-1}\mathbf{A}^{\cdot *}(\mathrm{E} - 2\varepsilon_e) + {}^1/_2(\mathrm{E} - 2\varepsilon_e)\mathbf{A}^{\cdot}\mathbf{A}^{-1} + \varphi \\ \varphi = (\boldsymbol{\Phi}_2{}^* - \boldsymbol{\Phi}_2)/2\end{gathered} \tag{5}$$

which agree with the equations describing the change in the strain tensor /1/, where they are obtained from other considerations. The method given here to obtain (5) yields the same results, but clarifies the meaning of the strain tensor ε introduced in /1/, constructed on the basis of the elastic part of the affinor $\mathbf{A}_e$. Eqs. (4) are equivalent to the relationships in /1/ in the domain of smoothness of the solutions but, unlike them, are in divergent form.

The matrix $\boldsymbol{\Phi}_2$, and therefore $\boldsymbol{\Phi}$ also, are not defined uniquely $\boldsymbol{\Phi}_2 = \boldsymbol{\varphi} + \boldsymbol{\varphi}_A$ where φ_A is any antisymmetric matrix.

The nature of the ambiguity is seen from the following consideration:

$$\mathbf{A}_p^{\cdot} = -\mathbf{A}_e{}^*(\boldsymbol{\varphi} + \boldsymbol{\varphi}_A)\mathbf{A}_e\mathbf{A}_p$$

If $\mathbf{A}_p = \mathbf{A}_p{}^\circ$ and $\mathbf{A}_e = \mathbf{A}_e{}^\circ$ at the time t_0, then, to the within higher-order infinitesimals, at the time $t_0 + \Delta t$

$$\mathbf{A}_p = -(\mathrm{E} + \mathbf{A}_e{}^*\boldsymbol{\varphi}_A\mathbf{A}_e\Delta t)(\mathrm{E} + \mathbf{A}_e{}^*\boldsymbol{\varphi}\mathbf{A}_e\Delta t)\mathbf{A}_p{}^\circ$$

The matrix $\mathrm{E} + \mathbf{A}_e{}^*\boldsymbol{\varphi}_A\mathbf{A}_e\Delta t$ corresponds to rotation of the element of the continuous medium as a whole. Hence, by virtue of the definitions of $\mathbf{A}_e$ and $\mathbf{A}_p$ we see that this ambiguity exerts no influence on the motion of the medium and, in particular, on the entropy production. The corresponding expression agrees with that presented in /1/. If it is considered that rotation of the element of the medium as a rigid body is referred entirely to $\mathbf{A}_e$, then $\mathbf{A}_p$ is a symmetric matrix.

We see that the plastic deformation process is performed without a change in volume or $\Delta_p = \det A_p = 1$.

Indeed, by definition

$$\Delta_p^{\cdot} = \dot{A}_{pij}\Delta_{pij} = \Delta_p \dot{A}_{pij} A_{pji}^{-1} = \Delta_p A_e A_e^* : \varphi = -\Delta_p G_e^{-1} : \varphi$$

The condition $\Delta_p^{\cdot} = 0$ is equivalent to the requirement of satisfying the continuity equation /1/, as is hence seen.

Now, when the equivalence of the equations obtained to the system of equations in /1/ is established, the results of /6, 7/ can be used for their closure, where semi-empirical equations of state and interpolation formulas of the kinetics of plastic deformation for a number of metals are presented.

The authors are grateful to G.I. Kanel' for drawing their attention to this problem.

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Translated by M.D.F.

PMM U.S.S.R., Vol.51, No.6, pp.761-769, 1987
Printed in Great Britain
0021-8928/87 $10.00+0.00

ASYMPTOTIC SOLUTION OF A QUASISTATIC THERMOELASTICITY PROBLEM FOR A SLENDER ROD*

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An asymptotic expansion is constructed for solving a quasistatic thermoelasticity problem for a slender cylindrical rod in the presence of mass forces and non-linear heat sources. The algorithm for constructing the asymptotic form, based on the method of boundary functions, is fairly simple and convenient for carrying out numerical calculations. A deduction is made on the basis of the asymptotic form constructed on how to select correctly a simplified one-dimensional model so as to obtain a better approximation for the solution of the initial two-dimensional problem. An existence theorem for the solution is proved under certain conditions.

1. Formulation of the problem. In the linear approximation the system of thermoelasticity equations for the displacement vector $u(x, y, z, t)$ and temperature $\theta(x, y, z, t)$ in a certain domain G has the form /1/

$$\begin{aligned} &\mu\Delta u + (\lambda + \mu)\,\mathrm{grad\,div}\, u + X = \gamma\,\mathrm{grad}\,\theta + \rho_0 u^{\cdot\cdot} \\ &\Delta\theta - \varkappa^{-1}\theta^{\cdot} - \eta\,\mathrm{div}\, u^{\cdot} = -\varkappa^{-1} H \end{aligned} \qquad (1.1)$$

*Prikl.Matem.Mekhan.,51,6,989-999,1987